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The Gaussian Inequality for Multicomponent Rotators

J. Bricmont[1]

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The Gaussian inequality is proven for multicomponent rotators with negative correlations between two spin components. In the case of one-component systems, the Gaussian inequality is shown to be a consequence of Lebowitz' inequality. For multicomponent models, the Gaussian inequality implies that the decay rate of the truncated correlation (or Schwinger) functions is dominated by that of the two-point function. Applied to field theory, these inequalities give information on the absence of bound states in the $\lambda(\phi_1^2 + \phi_2^2)^2$ model.

KEY WORDS: Classical rotators; correlation inequalities.

1. INTRODUCTION

Since it was introduced in Ising spin systems by Newman,[18] the Gaussian inequality had led to some interesting applications in statistical mechanics and quantum field theory. This inequality states that the correlation (or Schwinger) functions of an Ising system or a ϕ^4 theory are bounded by the correlation functions of a Gaussian (free) system with the same covariance.

Thus, the Gaussian inequality gives bounds on the n-point functions in terms of a bound on the two-point function.[10,18] It also allows one to recover results of Feldman[6] and Spencer[22] on the absence of even bound states of energy $<2m$ in a one-phase ϕ^4 theory.[18] The special case of the Gaussian inequality for the four-point function was discovered also by Lebowitz[14] as a consequence of a general inequality [(2.5a) in Ref. 14] that we shall call Lebowitz' inequality. This special case was applied to ϕ^4 in recent works of Glimm and Jaffe.[8,9]

Until now it was not known whether the Gaussian inequality was valid for multicomponent spin systems (and field theories). We show in this paper

[1] Institut de Physique Théorique, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

(Sections 2 and 3) how to derive the Gaussian inequality for multicomponent models from the generalized Griffiths inequalities introduced by Monroe[17] for two-component models and extended to three- and four-component models by Dunlop[2] and Kunz *et al.*[13]

On the other hand, the proof of the Gaussian inequality by Newman[18] as well as that by Sylvester[24] rely on combinatorial methods which are valid for Ising spin-1/2 systems. Then, by the classical Ising approximation,[12] the inequality is extended to ϕ^4 and other models. This class of models is quite close to the one for which Lebowitz' inequality is valid.[4,23] However, except for the four-point function, no relationship between the two inequalities was known. Using exactly the same method as with the multicomponent systems, we show (Sections 2 and 3) that the Gaussian inequality is really a consequence of Lebowitz' inequality.

In Section 4, we show that in the one-phase region (at zero external field) of a $(\phi_1^2 + \phi_2^2)^2$ theory or for the plane rotator model, the truncated correlation (or Schwinger) functions are dominated by the two-point function. This leads to an extension to two-component models of the spectral results of Spencer[22] ("absence of even bound states") based on Lebowitz' inequality and of those of Simon[20] ("coupling to the first excited state") based on the FKG inequality. Finally, the results of Mac Bryan and Spencer[16] on the decay rate of the spin–spin correlation function in the two-dimensional plane rotator model carry over to all the truncated correlation functions.[2]

2. THE GAUSSIAN INEQUALITY

The basic system in which we are interested is the following: we consider a family of D-dimensional random variables $\{\mathbf{s}_i | \mathbf{s}_i \in \mathbb{R}^D\}_{i\in\Lambda}$ indexed by the finite set Λ. Their joint probability distribution is

$$d\mu_\Lambda = Z_\Lambda^{-1} \exp \beta\Big(\sum_{i,j\in\Lambda} J_{ij}\mathbf{s}_i\cdot\mathbf{s}_j + \sum_{i\in\Lambda} \mathbf{h}_i\mathbf{s}_i\Big) \prod_{i\in\Lambda} d\nu_i(\mathbf{s}_i) \tag{1}$$

where $\nu_i(\mathbf{s}_i) = \rho_i(\|\mathbf{s}_i\|)\, d\mathbf{s}_i$ and are such that

$$\int \exp(b\|\mathbf{s}_i\|^2)\, d\rho_i(\|\mathbf{s}_i\|) < \infty \qquad \forall b\in\mathbb{R}, \quad \forall i\in\Lambda \tag{2}$$

Z_Λ is the normalization factor such that $\mu_\Lambda(\mathbb{R}^{D\cdot|\Lambda|}) = 1$.

This model is often considered in statistical mechanics and occurs as the lattice approximation of some field theories.[21]

[2] F. Dunlop has informed us that he has obtained very similar results (Theorems 2.1, 4.1, and 4.2 and their corollaries) in the cases $D = 1$ and 2, with slightly more general single-spin measures. His methods, which are quite different from ours, also give an extension of the Lee–Yang theorem.[25] We thank F. Dunlop for communicating his results to us before publication.

However, instead of considering this model directly, we shall state the theorems for measures that already satisfy certain correlation inequalities. This will make the proofs conceptually simpler. We shall also mention the cases where the model (1) is known to satisfy these inequalities.

Notations

$\mathbb{L}$ is a countable set.

$\Omega = (\mathbb{R}^D)^{\mathbb{L}}$.

For each $i \in \mathbb{L}$, $\mathbf{s}_i$ denotes the function on Ω that assigns to each configuration its value at i.

We denote by q_i, t_i the first two components of $\mathbf{s}_i$.

We also use the following functions:

$$x_i = (q_i + t_i)/\sqrt{2} \tag{3}$$

$$y_i = (q_i - t_i)/\sqrt{2} \tag{4}$$

Given a finite family of elements of $\mathbb{L}$ indexed by the set A, $\{i_a | i_a \in \mathbb{L}\}_{a \in A}$, we denote for any $B \subseteq A$

$$q_B = \prod_{a \in B} q_{i_a}$$

and similarly for t_B, x_B, y_B. If $B = \varnothing$, $q_B = 1$.

M is the set of finite families of elements of $\mathbb{L}$; each of these families is denoted by its set of indices.

Given a set E, we denote by a tilde the complementation in E ($\tilde{E}$).

We introduce now the correlation inequalities that our measures have to satisfy.

Definition 2.1. A measure on Ω has *negative correlations* if the expectations with respect to this measure satisfy, $\forall A, B \in M$.

$$\text{(i)} \quad 0 \leqslant \langle q_A t_B \rangle \leqslant \langle q_A \rangle \langle t_B \rangle \tag{5}$$

$$\text{(ii)} \quad 0 \leqslant \langle x_A y_B \rangle \leqslant \langle x_A \rangle \langle y_B \rangle \tag{6}$$

Remarks. (1) The measure (1), with $\mathbb{L} = \Lambda$, is known to have negative correlations when $D = 2, 3, 4$ provided that, $\forall i, j \in \Lambda$, $J_{ij} \geqslant 0$,

$$\mathbf{h}_i = (h_{i_1}, h_{i_2}, 0, 0), \qquad h_{i_1}, h_{i_2} \geqslant 0, \qquad h_{i_1} \geqslant h_{i_2}$$

and provided that ν_i satisfy certain conditions.[2,13] For example, the following measures satisfy these conditions:

$$d\nu_i(\mathbf{s}_i) = \delta(\|\mathbf{s}_i\| - a)\, d\mathbf{s}_i, \qquad a > 0 \tag{7}$$

$$d\nu_i(\mathbf{s}_i) = \exp(-a\|\mathbf{s}_i\|^4 + b\|\mathbf{s}_i\|^2)\, d\mathbf{s}_i, \qquad a > 0, \quad b \in \mathbb{R} \tag{8}$$

(2) If $D = 1$, let $s_i = q_i$ and consider $\mu_\Lambda(\{q_i\}) \times \mu_\Lambda(\{t_i\})$ as a measure on $(\mathbb{R}^2)^{|\Lambda|}$. If J_{ij}, h_i are nonnegative and if ν_i are even measures such that μ_Λ satisfies Lebowitz' inequality,[14,23] then $\mu_\Lambda \times \mu_\Lambda$ has negative correlations. Indeed, the relation (i) in Definition 2.1 is then an equality because of the factorization and relation (ii) is just Lebowitz' inequality. The positivity follows from the first Griffiths inequality.

Definition 2.2. A measure on Ω is (q, t)*-symmetric* if, for all A, $B \in M$,

$$\langle q_A t_B \rangle = \langle q_B t_A \rangle \tag{9}$$

In the model (1), this imposes the additional restriction, $\forall i \in \Lambda, h_{i_1} = h_{i_2} = h_i$. We call h_i the *external field.*

Definition 2.3 (see Ref. 18). A *pair-partition of* A, $A \in M$, is a partition of A, $\{B_i\}_{i\in I}$, such that if $|A|$ is even, $|B_i| = 2$, $\forall i \in I$. Or, if $|A|$ is odd, $|B_j| = 1$ for a single j and $|B_i| = 2$, $\forall i \neq j$.

Given a measure μ on Ω and $A \in M$,

$$P_\mu(A) = \sum_{p\in R(A)} \prod_{B_i \in p} \langle q_{B_i} \rangle \tag{10}$$

where $R(A)$ is the set of pair-partitions of A. (The subscript μ will be omitted if unnecessary.)

Theorem 2.1. If the measure μ on Ω has negative correlations and is (q, t)-symmetric, the expectations with respect to μ satisfy, $\forall A, B \in M$,

$$\langle q_A t_B \rangle \leqslant P_\mu(A) P_\mu(B) \tag{11}$$

The proof of this theorem relies on the following combinatorial lemma:

Lemma 2.1. Let f be a real-valued function on M such that, $\forall D \in M$,

$$f(D) = \sum_{p\in R(D)} \prod_{B_i \in p} f(B_i) \tag{12}$$

Then, if $A \in M$, $|A| = 2n$ or $|A| = 2n + 1$ and $k \leqslant n$,

$$\sum_{B \subseteq A;\, |B| = 2k} f(B) f(\tilde{B}) = \binom{n}{k} f(A) \tag{13}$$

We mention the following particular case of Theorem 2.1.

Corollary 2.1. Under the assumptions of Theorem 2.1, $\forall i, j, k, l \in \mathbb{L}$,

(i) $\langle q_i q_j q_k q_l \rangle \leqslant \langle q_i q_j \rangle \langle q_k q_l \rangle + \langle q_i q_k \rangle \langle q_j q_l \rangle + \langle q_i q_l \rangle \langle q_j q_k \rangle$ (14)

(ii) $\langle q_i q_j q_k \rangle \leqslant \langle q_i \rangle \langle q_j q_k \rangle + \langle q_j \rangle \langle q_k q_l \rangle + \langle q_k \rangle \langle q_j q_l \rangle$ (15)

Remarks. (1) Relation (11) contains the Gaussian inequality for $D = 1$,[18] which is thus a direct consequence of Lebowitz' inequality since $\mu_\Lambda \times \mu_\Lambda$ is by definition (q, t)-symmetric.

(2) When $h_1 = h_2 = 0$, (14) is the special case of the Gaussian inequality due to Lebowitz mentioned in the introduction. On the other hand, (15) resembles but is not as strong as the GHS inequality[11] for $D = 1$.

3. PROOF OF THE GAUSSIAN INEQUALITY

We start with the following result.

Proof of Lemma 2.1. Fix a term in the rhs of (12) with $D = A$ and call it T. Let p_T be the partition associated with T. The partition p_T is a product of $f(B_i)$, where B_i is a two-element set for all i, except possibly one, and for this one, $|B_i| = 1$. The number of factors in T with $|B_i| = 2$ is therefore n, since $|A| = 2n$ or $2n + 1$.

Let us compute how many times T will appear if we develop in all the terms of (13) $f(B)$ and $f(\tilde{B})$ according to (12). Now, T will appear as many times as we can choose k elements in p_T with $|B_i| = 2$; that is, exactly $\binom{n}{k}$ times. Indeed, choose B_i, $i = 1,..., k$, $B_i \in p_T$, and let $C = \bigcup_{i=1}^{k} B_i$. Since $|C| = 2k$, C occurs in the lhs of (13) and for this C, the development of $P(C)P(\tilde{C})$ gives T exactly once. Since the factor $\binom{n}{k}$ is the same for all the terms, the result follows. ∎

Proof of Theorem 2.1. Since, by Definitions 2.1 and 2.2, $\langle q_A t_B\rangle \leqslant \langle q_A\rangle\langle t_B\rangle = \langle q_A\rangle\langle q_B\rangle$, it is sufficient to prove

$$\langle q_A\rangle \leqslant P(A), \qquad \forall A \in M$$

We proceed by recurrence on $|A|$. Since the result is obvious for $|A| = 1$ and 2, we may assume it is true for all $B \in M$ with $|B| < |A|$. We distinguish two cases, depending on whether $|A|$ is even or odd.

(i) $|A| = 2n + 1$. By (3) and (4) we have

$$q_i = (x_i + y_i)/\sqrt{2} \tag{16}$$

$$t_i = (x_i - y_i)/\sqrt{2} \tag{17}$$

So we can express q_A in terms of x_i, y_i:

$$\langle q_A\rangle = 2^{-|A|/2} \sum_{C\subseteq A} \langle y_C x_{\tilde{C}}\rangle \leqslant 2^{-|A|/2}\Bigg(\sum_{\substack{\varnothing \neq C \subsetneq A \\ |C| \text{ even}}} \langle y_C\rangle\langle x_{\tilde{C}}\rangle + \langle x_A\rangle \Bigg) \tag{18}$$

The inequality comes from the negative correlations and the fact that, writing

y_C in terms of q_i and t_i, (q, t)-symmetry implies that $\langle y_C \rangle = 0$ if $|C|$ is odd. Now, for $C \in M$, $|C|$ even, $|C| < |A|$,

$$\langle y_C \rangle = 2^{-|C|/2} \sum_{E \subseteq C} (-)^{|E|} \langle t_E q_{\tilde{E}} \rangle$$

Since $\langle t_E q_{\tilde{E}} \rangle \geqslant 0$, we can drop the terms with $|E|$ odd:

$$\langle y_C \rangle \leqslant 2^{-|C|/2} \sum_{\substack{E \subseteq C \\ |E| \text{ even}}} \langle t_E \rangle \langle q_{\tilde{E}} \rangle \leqslant 2^{-|C|/2} \sum_{\substack{E \subseteq C \\ |E| \text{ even}}} P(E) P(\tilde{E})$$

Here, we have used the recurrence hypothesis ($|C| < |A|$). Using Lemma 2.1,

$$\sum_{\substack{E \subseteq C \\ |E| \text{ even}}} P(E)P(\tilde{E}) = \sum_{k=0}^{|C|/2} \sum_{\substack{E \subseteq C \\ |E| = 2k}} P(E)P(\tilde{E}) = \sum_{k=0}^{|C|/2} \binom{|C|/2}{k} P(C) = 2^{|C|/2} P(C)$$

Therefore

$$\langle y_C \rangle \leqslant P(C) \tag{19}$$

Considering x_C, $C \in M$, $|C|$ odd, $|C| < |A|$, we have

$$\begin{aligned}
\langle x_C \rangle &= 2^{-|C|/2} \sum_{I \subseteq C} \langle q_I t_{\tilde{I}} \rangle \\
&= 2 \cdot 2^{-|C|/2} \sum_{\substack{I \subseteq C \\ |I| \text{ even}}} \langle q_I t_{\tilde{I}} \rangle \qquad \text{by } (q, t)\text{-symmetry} \\
&\leqslant 2 \cdot 2^{-|C|/2} \sum_{\substack{I \subseteq C \\ |I| \text{ even}}} \langle q_I \rangle \langle t_{\tilde{I}} \rangle \qquad \text{by the negative correlations} \\
&\leqslant 2 \cdot 2^{-|C|/2} \sum_{\substack{I \subseteq C \\ |I| \text{ even}}} P(I)P(\tilde{I}) \qquad \text{by recurrence} \\
&= 2 \cdot 2^{-|C|/2} \sum_{k=0}^{(|C|-1)/2} \binom{(|C|-1)/2}{k} P(C) \\
&= 2^{1/2} P(C)
\end{aligned} \tag{20}$$

by Lemma 2.1.

Similarly for $\langle x_A \rangle$ we get, since $(|A| - 1)/2 = n$,

$$\begin{aligned}
\langle x_A \rangle &\leqslant 2 \cdot 2^{-|A|/2} \langle q_A \rangle + 2 \cdot 2^{-|A|/2} \sum_{k=1}^{n} \binom{n}{k} P(A) \\
&= 2^{-n+1/2} \langle q_A \rangle + 2^{-n+1/2} (2^n - 1) P(A)
\end{aligned} \tag{21}$$

Combining (18)–(21), we have

$$\langle q_A \rangle \leqslant 2^{-|A|/2} \Bigg[2^{1/2} \sum_{\substack{\varnothing \neq C \subsetneq A \\ |C| \text{ even}}} P(C)P(\tilde{C}) + 2^{-n+1/2}(2^n - 1)P(A) + 2^{-n+1/2} \langle q_A \rangle \Bigg]$$

Using again Lemma 2.1, we get

$$\frac{2^{2n}-1}{2^{2n}}\langle q_A\rangle \leqslant \frac{2^n-1}{2^n}P(A) + \frac{2^n-1}{2^{2n}}P(A) = \frac{2^{2n}-1}{2^{2n}}P(A)$$

(ii) $|A| = 2n$

$$\begin{aligned}\langle q_A\rangle &= 2^{-n}\Bigg(\sum_{\varnothing\neq C\subsetneq A}\langle x_C y_{\tilde C}\rangle + \langle x_A\rangle + \langle y_A\rangle\Bigg)\\ &\leqslant 2^{-n}\Bigg(\sum_{\substack{\varnothing\neq C\subsetneq A\\ |C|\text{ even}}}\langle x_C\rangle\langle y_{\tilde C}\rangle + \langle x_A\rangle + \langle y_A\rangle\Bigg)\end{aligned}\tag{22}$$

$$\sum_{\substack{\varnothing\neq C\subsetneq A\\ |C|\text{ even}}}\langle x_C\rangle\langle y_{\tilde C}\rangle = 2^{-n}\sum_{\substack{\varnothing\neq C\subsetneq A\\ |C|\text{ even}}}\sum_{\substack{E\subseteq C\\ I\subseteq \tilde C}}(-)^{|\tilde I|}\langle q_E t_{\tilde E}\rangle\langle q_I t_{\tilde I}\rangle\tag{23}$$

(in this sum $\tilde E$ is the complement of E in C, and $\tilde I$ is that of I in $\tilde C$)

$$\leqslant 2^{-n}\sum_{\substack{\varnothing\neq C\subsetneq A\\ |C|\text{ even}}}\sum_{\substack{E\subseteq C;\, I\subseteq\tilde C\\ |E|,\,|I|\text{even}}}\langle q_E\rangle\langle t_{\tilde E}\rangle\langle q_I\rangle\langle t_{\tilde I}\rangle$$

The last line comes from the fact that the terms in (23) with $|E|$, $|\tilde E|$, $|I|$, $|\tilde I|$ odd are all negative (due to the factor $(-)^{|\tilde I|}$) and that the sum over all the terms with $|E|$, $|\tilde E|$ odd and $|I|$, $|\tilde I|$ even is exactly the opposite of the sum over the terms with $|E|$, $|\tilde E|$ even and $|I|$, $|\tilde I|$ odd; so these two sums cancel each other.

Now, by recurrence and Lemma 2.1, the last sum is bounded by

$$\sum_{\substack{\varnothing\neq C\subsetneq A\\ |C|\text{ even}}} P(C)P(\tilde C) = (2^n-2)P(A)$$

We have also that

$$\begin{aligned}\langle x_A\rangle + \langle y_A\rangle &\leqslant 2\cdot 2^{-n}\Bigg(\sum_{\substack{\varnothing\neq C\subsetneq A\\ |C|\text{ even}}}\langle q_C\rangle\langle t_{\tilde C}\rangle + 2\langle q_A\rangle\Bigg)\\ &\leqslant 2\cdot 2^{-n}[(2^n-2)P(A) + 2\langle q_A\rangle]\end{aligned}$$

Therefore,

$$\langle q_A\rangle \leqslant 2^{-n}\{(2^n-2)P(A) + 2\cdot 2^{-n}[(2^n-2)P(A) + 2\langle q_A\rangle]\}$$

and

$$\begin{aligned}\frac{2^{2n-2}-1}{2^{2n-2}}\langle q_A\rangle &\leqslant \frac{2^n-2}{2^n}P(A) + \frac{2^n-2}{2^{2n-1}}P(A)\\ &= \frac{2^{2n-1}-2}{2^{2n-1}}P(A) \quad \blacksquare\end{aligned}$$

Remarks. (1) Neglecting the terms with a minus sign ($|E|$ odd) in the proof may be making a rough estimate when there is a large external field.

However, this estimate cannot be improved for an arbitrary external field, since these terms vanish with the external field.

(2) If we keep the neglected terms in the proof, we get, for example, for the three-point function,

$$\begin{aligned}\langle q_i q_j q_k\rangle \leqslant \langle q_i\rangle\langle q_j q_k\rangle + \langle q_j\rangle\langle q_i q_k\rangle \\ + \langle q_k\rangle\langle q_i q_j\rangle - \tfrac{2}{3}(\langle q_i\rangle\langle q_j t_k\rangle + \langle q_j\rangle\langle q_i t_k\rangle + \langle q_k\rangle\langle q_i t_j\rangle)\end{aligned}$$

Since $\langle q_i t_k\rangle \leqslant \langle q_i\rangle\langle q_k\rangle$, this is still a weaker inequality than the GHS inequality, except if $D = 1$, where it is the GHS inequality

$$(\langle q_i t_k\rangle = \langle q_i\rangle\langle q_k\rangle).$$

(3) If, instead of considering (q, t)-symmetric measures, we let the field be in the q direction (i.e., $h_{i_1} = h_i$, $h_{i_2} = 0$), we can get analogous bounds, for example,

$$\langle q_A\rangle \leqslant P(A) \qquad \text{if } |A| \text{ is odd}$$

$$\langle q_A + t_A\rangle \leqslant 2P(A) \qquad \text{if } |A| \text{ is even}$$

In fact, this can be seen from the proof, since putting the field in the q direction amounts to interchanging the roles of (q_i, t_i) and (x_i, y_i).

(4) We see that the model (1) with $D = 1$ and $h_i = 0$ is also (x, y)-symmetric. Therefore Theorem 2.1 applies, if the negative correlation holds, not only to the spin variable q_i, but also to the linear combinations x_i, y_i [(3), (4)] of the duplicate variables:

$$\langle x_A\rangle \leqslant \sum_{p\in R(A)} \prod_{B_i\in p} \langle x_{B_i}\rangle \tag{24}$$

Remark 3 deals also with this case if $h_i \neq 0$.

4. DOMINATION BY THE TWO-POINT FUNCTION

The goal of this section is to show that if the external field is zero in the model (1), one can bound the decay rate of all the truncated correlation functions by that of the two-point function.

Definition 4.1. A measure on Ω is *totally symmetric* if (i) $\forall A, B \in M$, $\langle q_A t_B\rangle = 0$ unless $|A|$ and $|B|$ are both even, and (ii) $\langle q_A\rangle = \langle t_A\rangle = \langle x_A\rangle = \langle y_A\rangle$.

In the model (1) this means that $\mathbf{h}_i = \mathbf{0}$ and $D > 1$. Our measure has to satisfy one more inequality:

Definition 4.2. A measure on Ω satisfies *Ginibre's inequality* if, $\forall A$, B, C, $D \in M$ with $|B|$, $|D|$ even and $e_i = \pm 1$,

$$\langle q_A(q_B + e_1 t_B) q_C(q_D + e_2 t_D)\rangle \geqslant \langle q_A(q_B + e_1 t_B)\rangle\langle q_C(q_D + e_2 t_D)\rangle \tag{25}$$

This is an unusual formulation of Ginibre's inequality,[7] and should rather be considered as a consequence of Ginibre's inequality.[1,2] This inequality is known[2,3,7] for the measure (1) with $D = 2, 3, 4$, $J_{ij} \geqslant 0$, $\mathbf{h}_i = (h_{i_1}, 0, 0, 0)$, $h_{i_1} \geqslant 0$, and ν_i given by (7) or (8) if $D = 3, 4$ or by any measure satisfying (2) if $D = 2$.

Given a distance on $\mathbb{L}$ (usually $\mathbb{L} = \mathbb{Z}^\nu$) and $A, B \in M$, we denote by

$$d(A, B) = \inf\{\operatorname{dist}(i_a, i_b) | a \in A, b \in B\}$$

$A \vee B$ is the disjoint union of A and B. Note that $q_A q_B = q_{A \vee B}$.

Theorem 4.1. If the measure μ is (q, t)-symmetric and has negative correlations, we have that, $\forall A, B \in M$,

$$\langle q_A q_B \rangle - \langle q_A \rangle \langle q_B \rangle \leqslant 2^{1-(|A|+|B|)/2} \sum_{\substack{C \subseteq A;\, D \subseteq B \\ |C|,\, |D| \text{ odd}}} \langle x_{\tilde{C}} x_{\tilde{D}} \rangle \langle y_C y_D \rangle \tag{26}$$

Theorem 4.2. Assume that μ is a totally symmetric measure on Ω having negative correlations and satisfying Ginibre's inequality. Assume also that there exists a distance on $\mathbb{L}$ and a nonincreasing function f on $[0, \infty[$ such that, $\forall i, j \in \mathbb{L}$,

$$\langle q_i q_j \rangle \leqslant f(\operatorname{dist}(i, j))$$

Then, for all $A, B, C, D \in M$ there exist constants K, K', K'' depending only on $|A|, |B|, |C|, |D|$ such that

$$\text{(i)} \qquad \langle q_A t_B q_C t_D \rangle \leqslant K f^2(d(A \vee B, C \vee D)) \tag{27}$$

if $|A|, |B|, |C|, |D|$ are odd;

$$\text{(ii)} \qquad \langle q_A t_B q_C t_D \rangle \leqslant K' f(d(A \vee B, C \vee D)) \tag{28}$$

if $|A|$ and $|C|$ (resp. $|B|$ and $|D|$) are odd and $|B|$ and $|D|$ (resp. $|A|$ and $|C|$) are even; and

$$\text{(iii)} \qquad |\langle q_A t_B q_C t_D \rangle - \langle q_A t_B \rangle \langle q_C t_D \rangle| \leqslant K'' f^2(d(A \vee B, C \vee D)) \tag{29}$$

if $|A|, |B|, |C|, |D|$ are even.

Proof of Theorem 4.1. By the hypotheses

$$\langle q_A q_B \rangle \geqslant \langle q_A \rangle \langle q_B \rangle = \langle q_A \rangle \langle t_B \rangle \geqslant \langle q_A t_B \rangle$$

Therefore

$$\begin{aligned} \langle q_A q_B \rangle - \langle q_A \rangle \langle q_B \rangle &\leqslant \langle q_A (q_B - t_B) \rangle \\ &= 2^{1-(|A|+|B|)/2} \sum_{\substack{C \subseteq A;\, D \subseteq B \\ |D| \text{ odd}}} \langle x_{\tilde{C}} y_C x_{\tilde{D}} y_D \rangle \end{aligned}$$

We have only terms with $|D|$ odd because of the minus sign in $(q_B - t_B)$. Then, by (q, t)-symmetry, the only nonvanishing terms are those with $|C|$ odd. The use of $\langle x_{\tilde{C}} y_C x_{\tilde{D}} y_D \rangle \leqslant \langle x_{\tilde{C}} x_{\tilde{D}} \rangle \langle y_C y_D \rangle$ concludes the proof. ∎

Proof of Theorem 4.2.

(i) By Theorem 2.1,

$$\langle q_A t_B q_C t_D \rangle \leqslant P(A \vee C) P(B \vee D)$$

Since $|A|$ and $|C|$ are odd, we have in each term of (10) applied to $P(A \vee C)$ at least one factor $\langle q_{i_a} q_{i_b} \rangle$ with $a \in A$ and $b \in C$. The same is true with $P(B \vee D)$. So there is, in each term of the rhs a product of two factors that are bounded by $f(d(A \vee B, C \vee D))$ since f is nonincreasing and $d(i_a, i_b) \geqslant d(A, C) \geqslant d(A \vee B, C \vee D)$. One can bound the other factors and the number of terms by a constant K.

(ii) The proof is similar. In general in each term we have only one factor bounded by $f(d(A \vee B, C \vee D))$ since $|A|$ and $|C|$ (or $|B|$ and $|D|$) are even.

(iii) We first use Ginibre's inequality to reduce the problem to the variables q_i. Indeed, one can show,[1,2] using (25), that

$$|\langle q_A t_B q_C t_D \rangle - \langle q_A t_B \rangle \langle q_C t_D \rangle| \leqslant \langle q_A q_B q_C q_D \rangle - \langle q_A q_B \rangle \langle q_C q_D \rangle \tag{30}$$

We use Theorem 4.1 and the fact that μ is totally symmetric to bound the rhs of (30) by

$$2^{1-(|A|+|B|+|C|+|D|)/2} \sum_{\substack{E \subseteq A \vee B \\ F \subseteq C \vee D \\ |E|, |F| \text{ odd}}} \langle q_E q_F \rangle \langle q_{\tilde{E}} q_{\tilde{F}} \rangle$$

Since $|A \vee B|$ and $|C \vee D|$ are even, $|\tilde{E}|$ and $|\tilde{F}|$ are odd. Application of Theorem 2.1 and the same method as in the proof of (i) concludes the proof. ∎

To conclude this paper, we discuss some applications based on the previous theorems and on results obtained elsewhere.[1,15,16,20,22]

Corollary 4.1. Let $\mathbb{L} = \mathbb{Z}^2$, with the usual distance, and consider the model given by (1), (7) with $\mathbf{h}_i = \mathbf{0}$, $J_{ij} \geqslant 0$, translation invariant and such that $J_{ij} = 0$ if $d(i, j) \neq 1$. Then, $\forall \epsilon > 0$, $\exists \beta_0$ such that $\forall \beta > \beta_0$, the conclusions of Theorem 4.2 hold for the translation-invariant equilibrium state of this model with the following function f: $f(x) = x^{-(1-\epsilon)/2\pi\beta}$.

Proof. The form of f is taken from Ref. 16, where the estimate is made on the two-point function with periodic boundary conditions. Since it was shown in Ref. 1 that this model has a unique invariant equilibrium state, the corollary follows from Theorem 4.2. ∎

One can use Theorem 4.2 to derive differentiability properties of the free energy and of the correlation functions with respect to β. In particular, if the two-point function is exponentially decreasing with dist(i, j), uniformly in some β-interval, the correlation functions and the free energy are C^∞ in β in the interior of this interval.

The following result is an application to quantum field theory. For the terminology we refer to Ref. 21.

Consider a field theory constructed (in two or three dimensions) with the interaction $\lambda(\phi_1^2 + \phi_2^2)^2 + \sigma(\phi_1^2 + \phi_2^2)$ using (half-) Dirichlet or (half-) periodic boundary conditions. Let Ω be the physical vacuum, H the renormalized Hamiltonian, and $\mathscr{H}$ the renormalized Hilbert space. Denote by $\mathscr{H}_e$ the subspace of $\mathscr{H}$ generated by

$$\left\{ \prod_{j=1}^{l} \prod_{i=1}^{n} \phi_1(f_i)\phi_2(f_j)|\Omega\rangle \middle| f_i, f_j \text{ are test functions and } l \text{ and } n \text{ are even} \right\}$$

and denote by $\mathscr{H}_0$ the subspace of $\mathscr{H}$ generated by

$$\left\{ \prod_{j=1}^{l} \prod_{i=1}^{n} \phi_1(f_i)\phi_2(f_j)|\Omega\rangle \middle| f_i, f_j \text{ are test functions and } l \text{ and } n \text{ are odd} \right\}$$

Corollary 4.2. With the preceding definitions:

(i) The vacuum is nondegenerate if and only if

$$\lim_{y\to\infty} \langle \phi_1(0)\phi_1(y)\rangle = 0$$

(ii) The spectrum of H lies outside $]0, m[$ if and only if, $\forall\epsilon > 0$,

$$\lim_{y\to\infty} \exp[(m - \epsilon)|y|] \langle \phi_1(0)\phi_1(y)\rangle = 0$$

(iii) If the spectrum of H lies outside $]0, m[$, then the spectrum of H restricted to the subspace $\mathscr{H}_e \oplus \mathscr{H}_0$ lies outside $]0, 2m[$.

Proof. Given Theorem 4.2, the corollary follows from general arguments. Points (i) and (ii) extend results of Simon[20,21] for one-component field theories and point (iii) extends a result of Spencer[22] and Feldman.[6] ∎

Apart from these corollaries, there exist other applications of the preceding theorems. We mention three of them:

1. The "mass gap" m in Corollary 4.2 is nondecreasing with σ. This is because the two-point function is nonincreasing with σ, by Ginibre's inequality.

2. Ellis and Newman[5] have shown that, when $D = 1$, the sign of Lebowitz' inequality is reversed in some models of the type (1) with certain single-spin measures ν_i. As a consequence of our method of proof, the sign of the Gaussian inequality (11) is also reversed for these measures, at least

in zero external field. These "reversed" inequalities also hold for two-component models (Newman, private communication).

3. If $D = 1$, $h_i = 0$, and the negative correlations (i.e., Lebowitz' inequality) hold, Theorem 4.1 also gives, using (24), the domination by the two-point function and the absence of even bound states of energy less than $2m$.[18,22] Indeed, $\langle x_i x_j \rangle = 2^{-1/2} \langle q_i q_j \rangle$.

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